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DEFLECTION BOUNDS FOR PLANE MEMBRANES AT MODERATE ROTATIONS

B. STORÅKERS
Department of Strength of Materials and Solid Mechanics, The Royal Institute of Technology, S-100 44 Stockholm, Sweden

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Abstract—For plane membranes deformed at small strains and moderate out-of-plane rotations, upper and lower bounds are derived for displacements conjugate to applied loads. Upper bounds are further given for displacements at arbitrary locations in static and dynamic circumstances extending an earlier approach by J. B. Martin to non-linear kinematics with rigor retained. The material properties considered are elastic, plastic, viscoelastic and viscoplastic. The bounding accuracy is examined by comparison of predicted deflections with those of some earlier and some apparently new exact solutions for particular cases involving contours of circular, annular and rectangular shapes.

INTRODUCTION

In the 1970s, the possibility of constructing a truly complementary energy principle applicable at finite deformation was discussed in depth by among others Zubov (1970), Christoffersen (1973), Ogden (1975) and Koiter (1976), the existence of such a principle having been indicated earlier by Hill (1956) and by Levinson (1965). Much of the attention was focused on the question, raised tentatively already by Levinson (1965), in what circumstances a relevant constitutive equation would admit unique inversion, a matter finally resolved for isotropic elastic solids "once and for all" by Ogden (1977).

After fundamental matters were settled, with few exceptions (Lee and Shield, 1980a, b), there seems to have been little interest in actually putting the new tool to work in conformity with classical methods for small deformation, that is to compute explicit though approximate results for particular cases with the principle as a base.

It was emphasized by Koiter (1976) that the complementary principle would be particularly well suited to deal with problems of membranes subjected to small strains and moderate rotations and in this setting Koiter (1976) derived and solved the exact field equations for a Hookean circular membrane under uniform lateral pressure. In this spirit, Stumpf in a series of papers (Stumpf, 1979), has specialized the principle to apply to von Karman plate theory and Marguerre shell theory and also in one case (Stumpf, 1975) obtained explicit energy bounds for rectangular plates. More recently the principle was applied by Storåkers (1983a) to plane Hookean membranes in order to obtain approximate results for some boundary contours and also upper bounds to the volume enclosed by the deformed and initial membrane shapes. Of more interest from a practical point of view, as regards the latter matter, is perhaps bounds for local deflections. It is the present purpose to show how such bounds may be derived in various static and dynamic circumstances and also demonstrate their application. Several popular constitutive material models will be considered.

THE BOUNDARY VALUE PROBLEM

Field equations

Attention will be confined to initially plane membranes having local thickness t and subjected to prescribed distributed pressure p_α, p_3 on the surface area S. Tractions T^0_α, T^0_3 and conjugate displacements u^0_α, u^0_3 are prescribed on parts Γ_T and Γ_u of the membrane contour respectively or alternatively in some mixed–mixed version. Greek indices then take on values 1 and 2 and refer to in-plane coordinates while subscript 3 refers to the transverse direction.

As a basic strain measure the reduced form of Green strain

$$e_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha} + u_{3,\alpha}u_{3,\beta}) \tag{1}$$

is adopted, where a comma denotes differentiation with respect to in-plane coordinates x_α. In eqn (1) squares of in-plane displacement gradients have been neglected in comparison with out-of-plane rotations. Thus for consistency, strains have to be small in the ordinary sense as do squares of rotations, a setting consistent with assumptions descending from Föppl (1907).

Postponing for a moment consideration of materials exhibiting viscous effects, to be dealt with below via corrrespondence principles, it is assumed that the membrane material in question possesses a strain energy function $W(e_{\alpha\beta})$ generating the second Piola–Kirchhoff stress tensor

$$\tau_{\alpha\beta} = \frac{\partial W}{\partial e_{\alpha\beta}}. \tag{2}$$

Towards adopting a stress measure independent of deformation, the nominal (or transpose of the first Piola–Kirchhoff) stress must necessarily be relied upon. Being conjugate to displacement gradients, this stress measure is then generated by

$$s_{\alpha i} = \frac{\partial W}{\partial u_{i,\alpha}} \tag{3}$$

the undeformed state being the reference configuration. In eqn (3) and onwards Latin indices range from 1 to 3.

Combination of eqns (1) and (3) yields

$$s_{\alpha\beta} = \frac{\partial W}{\partial e_{\alpha\beta}} \tag{4}$$

$$s_{\alpha 3} = \frac{\partial W}{\partial e_{\alpha\beta}} u_{3,\beta}. \tag{5}$$

From a virtual work principle

$$\oint T_i \delta u_i \,\mathrm{d}\Gamma + \int p_i \delta u_i \,\mathrm{d}S = \int s_{\alpha i} \delta u_{i,\alpha} t \,\mathrm{d}S \tag{6}$$

there follows the linear equilibrium equations

$$(ts_{\alpha i})_{,\alpha} + p_i = 0 \tag{7}$$

and dynamic boundary conditions

$$ts_{\alpha i} n_\alpha = T_i^0 \tag{8}$$

on Γ_T, n_α being the outwards unit normal to the contour.

Thus the field equations, eqns (1), (4), (5) and (7), together with boundary conditions (8) and displacement boundary conditions

$$u_i = u_i^0 \tag{9}$$

on Γ_u, constitute the boundary value problem to be solved.

The problem so formulated constitutes a consistent first-order approximation, a matter earlier being dealt with in detail by Hill and Storåkers (1980). Of particular interest, from a practical point of view, is that the distinction between uni-directional lateral loading, such as caused by gravity, and hydrostatic pressure loading in the ordinary sense, vanishes.

Having extremum principles in mind, and also uniqueness and stability, there is reason to dwell on requirements for convexity of the strain energy function W. To this end, when as presently adopting displacement gradients as primary variables, for strict convexity to prevail, it is required that

$$\frac{\partial^2 W}{\partial u_{i,\alpha}\,\partial u_{j,\beta}}\delta u_{i,\alpha}\,\delta u_{j,\beta} > 0 \tag{10}$$

for all $\delta u_{i,\alpha} \neq 0$.

By the strain–displacement relations (1), inequality (10) may be transformed to read

$$\frac{\partial^2 W}{\partial e_{\alpha\beta}\,\partial e_{\gamma\kappa}}(\delta u_{\alpha,\beta}\delta u_{\gamma,\kappa} + u_{3,\alpha}u_{3,\gamma}\delta u_{3,\beta}\delta u_{3,\kappa}) + \frac{\partial W}{\partial e_{\alpha\beta}}\delta u_{3,\alpha}\delta u_{3,\beta} > 0. \tag{11}$$

It is evident from the first terms in inequality (11) that convexity of W with respect to Green strain is required, which in the present setting leads to conventional constitutive requirements pertinent to linear kinematics. The last term in inequality (11) imposes the further condition that, by eqn (2), the stress tensor $\tau_{\alpha\beta}$ or, equivalently by eqn (4), $s_{\alpha\beta}$ should be positive definite. Thus only positive principal stresses are admissible.

Formally then the unloaded state is inadmissible. This pathological situation is not a real issue though as it derives from the vanishing of the initial transverse membrane stiffness. A purely one-dimensional case, string problem, may of course be treated on its own merits. Provided that the resulting conditions are fulfilled, it is readily shown by standard methods that the boundary value problem possesses a unique and stable solution. The converse is not necessarily true though.

Furthermore, by appeal to convexity and standard arguments, it is obvious that the solution to the boundary value problem formulated is governed by an analytical minimum of the potential

$$U = \int Wt\,\mathrm{d}S - \int p_i u_i\,\mathrm{d}S - \int T_i u_i\,\mathrm{d}\Gamma_T \tag{12}$$

and also simultaneously by a minimum of a complementary potential

$$\bar{U} = \int \bar{W}t\,\mathrm{d}S - \int T_i u_i\,\mathrm{d}\Gamma_u \tag{13}$$

provided that a strain energy function $\bar{W}$ complementary to W exists. This in turn requires unique invertibility of the constitutive equation.

Particular constitutive properties

Having commonly employed constitutive equations in mind, there does not seem to be a great loss of generality in assuming that the strain energy function W is homogeneous of degree $m+1$, say, in the strain components. Then by using eqn (2) and Euler's theorem for homogeneous functions

$$\tau_{\alpha\beta}e_{\alpha\beta} = (m+1)W. \tag{14}$$

Particular examples of such W are the Hookean (semi-linear) material ($m = 1$) with

$$W = \frac{E}{2(1+\nu)}\left(e_{ij}e_{ij} + \frac{\nu}{1-2\nu}e_{ii}e_{jj}\right) \tag{15}$$

in customary notation and three-dimensional form, and a power-law material with

$$W = \frac{\sigma_0}{m+1}\bar{e}^{m+1} \tag{16}$$

where σ_0 is a material constant and $\bar{e}$ some appropriate equivalent strain measure.

Now by using eqns (1), (2), (4) and (5), the constitutive relation (14) may be expressed as

$$s_{\alpha\beta}u_{\beta,\alpha} + \tfrac{1}{2}s_{\alpha 3}u_{3,\alpha} = (m+1)W. \tag{17}$$

Then by a Legendre transformation

$$W + \bar{W} = s_{\alpha i}u_{i,\alpha} \tag{18}$$

a complementary strain energy function $\bar{W}$ may be introduced, being explicitly

$$\bar{W} = \frac{m}{m+1}s_{\alpha\beta}u_{\beta,\alpha} + \frac{2m+1}{2(m+1)}s_{\alpha 3}u_{3,\alpha} \tag{19}$$

or alternatively by eqns (1), (2), (4) and (5)

$$\bar{W} = \frac{m}{m+1}\tau_{\alpha\beta}e_{\alpha\beta} + \tfrac{1}{2}s_{\alpha 3}u_{3,\alpha}. \tag{20}$$

The first term on the right-hand side of eqn (20) corresponds to a complementary strain energy function in the case of plane stress and linear kinematics. Provided that the constitutive equation in question is invertible in such a situation, it is also so in the present formulation as by eqns (4) and (5), explicitly

$$u_{3,\alpha} = \frac{\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}s_{\beta\delta}s_{\gamma 3}}{\varepsilon_{\alpha\beta}s_{\alpha 1}s_{\beta 2}} \tag{21}$$

where $\varepsilon_{\alpha\beta}$ is the two-dimensional permutation tensor.

By the convexity condition, inversion (21) is nonsingular and unique, the exception of the unloaded state having already been commented upon above. On inserting eqn (21) into eqn (19), the complementary strain energy may then be expressed solely by using the nominal stresses.

DISPLACEMENT BOUNDS

Static load-conjugate bounds

As they stand, the two potentials, eqns (12) and (13), introduced above are clearly of value as tools in particular situations for the derivation of approximate solutions to specific variables and also for the determination of energy release rates in the case of moving boundaries such as at delamination. With the view of obtaining local displacement bounds, however, it is desirable to attempt to express the internal strain energy explicitly in terms of the potential of external loads. There is reason to believe though that such a procedure may be carried out successfully at finite deformation only in particular circumstances.

In the present case, however, by eqn (17) and the divergence theorem combined with the equilibrium equations (7)

$$\int Wt\,\mathrm{d}S = \frac{1}{m+1}\left[\oint (T_\alpha u_\alpha + \tfrac{1}{2}T_3 u_3)\,\mathrm{d}\Gamma + \int (p_\alpha u_\alpha + \tfrac{1}{2}p_3 u_3)\,\mathrm{d}S\right] \tag{22}$$

and similarly by eqn (19)

$$\int \bar{W}t\,\mathrm{d}S = \frac{m}{m+1}\left[\oint\left(T_\alpha u_\alpha + \frac{2m+1}{2m}T_3 u_3\right)\mathrm{d}\Gamma + \int\left(p_\alpha u_\alpha + \frac{2m+1}{2m}p_3 u_3\right)\mathrm{d}S\right]. \tag{23}$$

Then remembering the minimum properties of the potentials introduced above, the defining equations, eqns (12) and (13), may be combined with eqns (22) and (23) to yield

$$\bar{U}(s^*_{\alpha i}) \geqslant \frac{m}{m+1}\left[\oint\left(T_\alpha u_\alpha + \frac{2m+1}{2m}T_3 u_3\right)\mathrm{d}\Gamma + \int\left(p_\alpha u_\alpha + \frac{2m+1}{2m}p_3 u_3\right)\mathrm{d}S\right] \geqslant -U(u_i^*) \tag{24}$$

when no external work is performed on part of the contour Γ_u and similarly

$$U(u_i^*) \geqslant \frac{1}{m+1}\left[\oint (T_\alpha u_\alpha + \tfrac{1}{2}T_3 u_3)\,\mathrm{d}\Gamma + \int (p_\alpha u_\alpha + \tfrac{1}{2}p_3 u_3)\,\mathrm{d}S\right] \geqslant -\bar{U}(s^*_{\alpha i}) \tag{25}$$

as regards Γ_T. In inequalities (24) and (25), $s^*_{\alpha i}$ and u_i^* are equilibrated stress and kinematically admissible displacement fields, respectively, which are in no way related. Thus if in the two common situations when inequalities (24) and (25) apply, a single external load or displacement agency is acting, double-sided bounds may be obtained for the work-conjugate variable.

Static upper bound for displacements at an arbitrary location

In fully linear problems the derivation of local displacement bounds for a body at arbitrary loading, in general rests on the existence of a reciprocity relation, which however is not available in the present setting. A similar situation was encountered by Martin (1966) when dealing with a kinematically linear but materially non-linear case. Martin's approach will then be adopted here with full account taken of the kinematic nonlinearity.

Thus as a consequence of the strict convexity properties of W

$$W(u_{i,\alpha}) - W(u^*_{i,\alpha}) \geqslant s^*_{i\alpha}(u_{i,\alpha} - u^*_{i,\alpha}) \tag{26}$$

where $s^*_{\alpha i}$, u_i^* are any fields constitutively admissible and thus related by eqn (3).

Then by eqn (18)

$$W(u_{i,\alpha}) + \bar{W}(s^*_{\alpha i}) \geqslant s^*_{\alpha i}u_{i,\alpha}. \tag{27}$$

By now choosing $s^*_{\alpha i}$ as a field in equilibrium with external loads T_i^*, p_i^*, application of the virtual work principle and eqn (22) yields

$$\int \bar{W}(s^*_{\alpha i})t\ \mathrm{d}S \geqslant \oint \left\{ \left(T^*_\alpha - \frac{1}{m+1} T_\alpha \right) u_\alpha + \left[T^*_3 - \frac{1}{2(m+1)} T_3 \right] u_3 \right\} \mathrm{d}\Gamma$$
$$+ \int \left\{ \left(p^*_\alpha - \frac{1}{m+1} p_\alpha \right) u_\alpha + \left[p^*_3 - \frac{1}{2(m+1)} p_3 \right] u_3 \right\} \mathrm{d}S. \quad (28)$$

On interpreting the starred load variables in inequality (28) as constituting a dummy load system, it is possible to bound from above the actual displacement u_i at an arbitrary location, $\bar{x}_\alpha$, provided displacements are prescribed to vanish on Γ_u. Thus for a particular displacement component $u_i(\bar{x}_\alpha)$, on choosing the dummy field as a colinear discrete force $P_i^*(\bar{x}_\alpha)$ combined with additional tractions such that the remaining integrals in inequality (28) vanish, the bound reduces to

$$u_i(\bar{x}_\alpha) \leqslant \frac{1}{P_i^*(\bar{x}_\alpha)} \int \bar{W}(s^*_{\alpha i})t\ \mathrm{d}S. \quad (29)$$

For the particular case when the dummy force is colinear with a single external load, inequality (29) is in conformity with inequality (24) and only in this situation may an equality sign in general be expected to prevail in inequality (29).

It is appropriate to mention in this context that, in the spirit of Martin, a corresponding lower bound has been derived by Palmer (1967) for linear kinematics. The resulting bounding inequality is more complex though and relies on both dynamic and kinematic trial fields. As moreover in particular cases explicit results appear to be fairly sensitive to the choice of trial fields, the matter is not elaborated upon further here.

Dynamic upper bound for displacements at arbitrary location

The dynamic situation to be considered concerns a membrane, which is externally unloaded and fixed on part Γ_u of its contour. The membrane is subjected to an impulse resulting in an instantaneous kinetic energy K_0. An upper bound for the displacement at an arbitrary location is sought for the ensuing motion during which it is assumed that convexity as above prevails.

Drawing upon Martin's (1964) approach to a fully linear elastic situation, at a particular instant when the kinetic membrane energy equals K, then for energy balance to prevail

$$K + \int Wt\ \mathrm{d}S = K_0. \quad (30)$$

Introducing inequality (27) and applying the divergence theorem yields

$$K_0 - K + \int \bar{W}(s^*_{\alpha i})t\ \mathrm{d}S \geqslant \oint T_i^* u_i\ \mathrm{d}\Gamma + \int p_i^* u_i\ \mathrm{d}S. \quad (31)$$

The starred field in inequality (31) is then preferably chosen as one in static equilibrium with a discrete dummy force, $P_i^*(\bar{x}_\alpha)$, colinear with the displacement component $u_i(\bar{x}_\alpha)$ sought for. As furthermore the kinetic energy K at any instant is necessarily nonnegative, inequality (31) may be weakened to read

$$u_i(\bar{x}_\alpha) \leqslant \frac{1}{P_i^*(\bar{x}_\alpha)} \left[K_0 + \int \bar{W}(s^*_{\alpha i})t\ \mathrm{d}S \right]. \quad (32)$$

As it stands inequality (32) provides an upper bound for displacements at any instant during the motion.

The bound just derived is formally equivalent to its counterpart in its simplest form in case of linear kinematics and the technique has in fact been applied to finite deformation earlier by Martin (1968) for linear elastic materials and Martin and Ponter (1972) for rigid/perfectly-plastic materials. Guided by physical insight, these writers have justified the application of the bounding technique by accompanying qualitative requirements that the body must behave in a stable manner under the action of dummy loads. As the applications dealt with concern situations when rotations are moderate, as in the present setting, the prerequisites may be made a little bit more precise.

As already discussed above, from a strictly mathematical point of view, an additional requirement of structural stability might not suffice but only positive in-plane principal stresses will guarantee the existence of a non-singular convex complementary strain energy function. It seems fairly obvious that the same should be true also for members such as beams and plates. The point may be proved rigorously, at least for Hookean materials, by drawing on explicit strain energy expressions for plates derived by Stumpf (1975). Furthermore, in the case of perfectly plastic material behavior, a complementary strain energy function may be formally introduced by deleting the first term on the right-hand side of eqn (20). The resulting expression then holds good for plates as well with only membrane stresses accounted for. Briefly then, in applications dealt with earlier the circumstances seem to have been such that the approach may be justified.

SELF-SIMILARITY AND CORRESPONDENCE PRINCIPLES

Of major interest in the present context are membranes subjected to pure transverse loading. It proves advantageous then to demonstrate first that the solutions for relevant field variables are self-similar in rather general circumstances. For uniform pressure loading and a circular membrane of power-law material, the matter was dealt with by Ilyushin (1956) and later in detail by Hill and Storåkers (1980) for arbitrary contours. Only a sketchy proof will be given then for the present slight generalization.

The particular loading conditions singled out for attention are

$$\left.\begin{array}{lll} p_\alpha = 0, & p_3 = \lambda f(x_\alpha) & \text{on} \quad S \\ u_\alpha = 0, & T_3 = \lambda g(x_\alpha) & \text{on} \quad \Gamma_T \\ u_i = 0 \quad \text{on} \quad \Gamma_u & & \end{array}\right\} \tag{33}$$

where λ is a scalar load parameter and f and g arbitrary functions characterizing the load distribution.

To this end it is advantageous to introduce dimensionless field variables $\tilde{u}_i$, $\tilde{e}_{\alpha\beta}$, $\tilde{s}_{\alpha i}$, which depend on dimensionless coordinates $\tilde{x}_\alpha = x_\alpha/l$ solely, l being a characteristic in-plane dimension. By further writing the strain energy function, homogeneous of degree $m+1$, in the form

$$W = \sigma_0 W'(e_{\alpha\beta}) \tag{34}$$

and introducing the particular ansatz

$$\left.\begin{array}{r} \dfrac{u_\alpha}{l\tilde{u}_\alpha} = \left(\dfrac{u_3}{l\tilde{u}_3}\right)^2 = \dfrac{e_{\alpha\beta}}{\tilde{e}_{\alpha\beta}} = \left(\dfrac{s_{\alpha\beta}}{\sigma_0 \tilde{s}_{\alpha\beta}}\right)^{1/m} = \lambda^{2/(2m+1)} \\ \dfrac{s_{\alpha\beta}}{\sigma_0 \tilde{s}_{\alpha\beta}} = \lambda \end{array}\right\} \tag{35}$$

it may be proved that the problem admits a solution separable in load and space variables. Thus by inserting eqn (35) into the relevant field equations and boundary conditions, it is readily seen that the dimensionless variables depend only on the relative load distribution and the geometry of the boundary.

Computational simplifications apart, eqn (35) implies proportional straining of material elements, which has the further virtue of justifying the use of deformation theory, when dealing with plastic materials under monotonic loading.

The presence of self-similarity has further consequences when considering time-dependent material properties as has been shown earlier for uniform pressure loading by Hill and Storåkers (1980) for non-linear creep and viscoplasticity and Storåkers (1983b) for linear viscoelasticity. Thus correspondence principles may be derived, which admit solutions separable in time and space.

A common way to simulate inelastic time-dependent behavior of metals is via some dissipation potential

$$\phi = \frac{h(\tau_{ij})}{k(e_{kl})} \tag{36}$$

generating strain rates

$$\frac{\partial e_{ij}}{\partial t} = \frac{\partial \phi}{\partial \tau_{ij}}. \tag{37}$$

If h is homogeneous of degree $(n+1)/n$ and k of degree q, say, then common constitutive equations, which fall into this category, are Norton's law for secondary creep, when h is a power function of some equivalent stress, and Nadai's law for primary (strain-hardening) creep when furthermore k is a power function of some equivalent strain. The latter version is also frequency utilized to simulate strain rate dependent plastic effects.

For this case it proves suitable to set up a trial solution of the form

$$\left.\begin{aligned} \frac{u_\alpha}{l\tilde{u}_\alpha} = \left(\frac{u_3}{l\tilde{u}_3}\right)^2 &= c \\ \left(\frac{s_{\alpha\beta}}{\sigma_0 \tilde{s}_{\alpha\beta}}\right) = \left(\frac{\mathrm{d}c}{\mathrm{d}t}\right)^n & c^{nq} \\ \frac{s_{\alpha 3}}{\sigma_0 \tilde{s}_{\alpha 3}} = \lambda & \end{aligned}\right\} \tag{38}$$

where $c(t)$ is a function of natural time sought for and $\lambda(t)$, as above though time dependent, is proportional to the external loading.

For the case of uniform pressure loading, it was found by Hill and Storåkers (1980) that an ordinary differential equation

$$\left(\frac{\mathrm{d}c}{\mathrm{d}t}\right)^n c^{nq+1/2} = \lambda \tag{39}$$

results for c. It is not difficult to show that the same holds true also here and any details are refrained from.

The matter was dealt with likewise by Storåkers (1983b) for the case of a linear viscoelastic constitutive equation

$$e_{\alpha\beta} = \int_0^t c_{\alpha\beta\gamma\delta}(t-\tau)\frac{\partial \tau_{\gamma\delta}}{\partial \tau}\,\mathrm{d}\tau \tag{40}$$

with creep compliances assumed to be of uniform time dependence such that

$$\underset{\sim}{c} = \underset{\sim}{c}_0 s(t) \tag{41}$$

say.

Thus retaining the trial solution for displacements according to eqns (38), an integral equation

$$c = \int_0^t s(t-\tau) \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\lambda}{\sqrt{c}}\right) \mathrm{d}\tau \tag{42}$$

results.

As a consequence of the existence of the correspondence principles just outlined, any bound derived in static circumstances will retain its validity for a corresponding time-dependent material when constitutive parameters are properly interpreted.

SOME EXACT SOLUTIONS FOR ANNULAR MEMBRANES

Membranes being fixed to an external circular contour are of obvious practical interest. In the case of an annular membrane, transversely loaded by means of a concentric rigid boss, apparently Schwerin (1929) was the first, among several, to discover that an axisymmetric solution in closed form exists in the case of a Hookean material having the particular Poisson's ratio $\nu = 1/3$ and constant thickness.

Thus in this case, with a strain energy function according to eqn (15) in its proper plane stress form, the constitutive relations reduce to

$$\left.\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}r} &= (s_{rr} - \nu s_{\phi\phi})/E - (s_{rz}/s_{rr})^2/2 \\ \frac{u}{r} &= (s_{\phi\phi} - \nu s_{rr})/E \\ \frac{\mathrm{d}w}{\mathrm{d}r} &= s_{rz}/s_{rr} \end{aligned}\right\} \tag{43}$$

when referred to a polar coordinate system, in obvious notation, and considering only axisymmetric cases.

The non-trivial equilibrium equations are, in the absence of distributed loading

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}(r s_{rr}) - s_{\phi\phi} &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}r}(r s_{rz}) &= 0 \end{aligned}\right\}. \tag{44}$$

At the external boundary $(r = a)$ the boundary conditions are

$$u = w = 0 \tag{45}$$

and at the internal boundary $(r = \kappa a)$

$$u = 0, \qquad s_{rz} = -P/(2\pi\kappa a t) \tag{46}$$

where P is the total transverse load.

It is then readily shown that, in particular when $\nu = 1/3$, the so formulated problem admits a solution of the form

$$\left.\begin{aligned} u &= 0 \\ w &= c_1[1-(r/a)^{\alpha}]a \\ s_{rr} &= s_{\phi\phi}/(1-\beta) = c_2(r/a)^{-\beta} \\ s_{rz} &= -P/(2\pi rt) \end{aligned}\right\} \tag{47}$$

where $\alpha = \beta = 2/3$ and

$$c_1^2 = 4c_2/E = 3^{2/3}\left(\frac{P}{\pi Eat}\right)^{2/3}.$$

The simplicity of this particular solution derives from the fact that the radial displacement component vanishes identically. It would seem natural then to investigate whether similar circumstances prevail for an incompressible power-law material with a strain–energy function as given by eqn (16).

Choosing an equivalent strain

$$\bar{e}^2 = \tfrac{4}{3}(e_{\alpha\beta}e_{\alpha\beta} + e_{\alpha\alpha}e_{\beta\beta}) \tag{48}$$

appropriate for plane stress, the constitutive relations corresponding to eqns (43) are

$$\left.\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}r} &= (\bar{s}/\sigma_0)^{1/m-1}(s_{rr}-s_{\phi\phi}/2)/\sigma_0 - (s_{rz}/s_{rr})^2/2 \\ \frac{u}{r} &= (\bar{s}/\sigma_0)^{1/m-1}(s_{\phi\phi}-s_{rr}/2)/\sigma_0 \\ \frac{\mathrm{d}w}{\mathrm{d}r} &= s_{rz}/s_{rr} \end{aligned}\right\} \tag{49}$$

where the associated equivalent stress $\bar{s}$ is defined by

$$\bar{s}^2 = s_{rr}^2 - s_{rr}s_{\phi\phi} + s_{\phi\phi}^2. \tag{50}$$

It is not difficult to show then that also for these material properties, a solution of the same form as eqns (47) exists for the particular value $m = 1/2$.

Thus in this case $\alpha = \beta = 1/2$ and

$$c_1 = 3^{3/4}c_2/\sigma_0 = 3^{3/8}\left(\frac{P}{\pi\sigma_0 at}\right)^{1/2}. \tag{51}$$

The particular choice of equivalent stress, eqn (50), corresponds to that of von Mises and remembering the proportionality of straining as discussed above, in addition to non-linear elastic materials, the solution is also valid for plastic strain-hardening materials obeying incremental flow theory and having the particular strain-hardening index $m = 1/2$.

It turns out, however, that when adopting Tresca's flow potential and its associated flow rule, the solution to the problem may be expressed in closed form for arbitrary values of m. Thus in this case the constitutive relations are

$$\left.\begin{aligned} \frac{du}{dr} &= (s_{rr}/\sigma_0)^{1/m} - (s_{rz}/s_{rr})^2/2 \\ u &= 0 \\ \frac{dw}{dr} &= s_{rz}/s_{rr} \end{aligned}\right\} \tag{52}$$

when $s_{rr} > s_{\phi\phi} > 0$.

For this case then in the notation of eqns (47)

$$\alpha = \beta = \frac{2m}{2m+1} \tag{53}$$

$$c_1 = \frac{2m+1}{\sqrt{2m}}(c_2/\sigma_0)^{1/2m} = \frac{2m+1}{\sqrt{2m}} \cdot 2^{-3/(2(2m+1))}\left(\frac{P}{\pi\sigma_0 at}\right)^{1/(2m+1)}. \tag{54}$$

In the particular case $m = 1/2$, eqn (54) reduces to eqn (51) if σ_0 in eqn (54) is replaced by $(2/\sqrt{3})^{3/2}\sigma_0$, as would be expected.

In the perfectly plastic limit, i.e. when $m \to 0$, the Tresca solution reduces to

$$\left.\begin{aligned} u &= 0 \\ w &= -\frac{P}{2\pi\sigma_0 t}\ln(r/a) \\ s_{rr} &= s_{\phi\phi} = \sigma_0 \\ s_{rz} &= -P/(2\pi r t) \end{aligned}\right\}. \tag{55}$$

For a vanishing boss radius it is evident that the solution in this case becomes singular. Furthermore, the solution might not be unique as it corresponds to a singular point on the yield surface. Also it should be observed that neither of the solutions above may formally be adopted to apply to the case of a point load as then, pointwise in the center of the membrane, the rotation is not moderate but infinite.

It is believed though that, apart from the present purpose, the solutions obtained may be helpful when analyzing some kinds of axisymmetric sheet metal forming problems. Such problems are initially severely non-linear and, as an alternative to guessing initial values in a numerical finite strain solution, such may be provided by the solutions discussed and also by some approximate ones below.

ILLUSTRATIONS OF BOUNDS

Static load-conjugate bound

The annular membrane subjected to transverse loading, as described above, is chosen to illustrate the load-conjugate bounding technique. For a Hookean material of arbitrary Poisson's ratio, two approximate solutions by Allman (1982) are already available. To derive these, Allman utilized two variation principles, one based solely on variation of displacements, corresponding to potential (12) above, and a mixed one. Guided by the existence of the analytical solution for $\nu = 1/3$, Allman chose fields in conformity with eqns (47) above with $\alpha = \beta = 2/3$ at the outset. The result for the non-dimensional deflection amplitude c_1, was

$$c_1 = \frac{3}{2}\left[\frac{(1-\nu^2)P}{\pi Eat}\right]^{1/3} \tag{56}$$

based on displacement variations and

$$c_1 = \frac{3}{2}\left[\frac{2(5-3\nu)P}{9\pi Eat}\right]^{1/3} \tag{57}$$

based on variation of in-plane stresses and transverse displacement.

It may be readily shown that the latter result is in conformity with variation of the complementary stress-based potential $\bar{U}$, eqn (13), and furthermore by inequalities (24) that eqns (56) and (57) constitute lower and upper bounds, respectively. The bounds reproduce the actual solution very closely as for $0.17 < \nu < 0.48$, the relative difference is less than 1%, the maximum difference being 3.6% for $\nu = 0$.

It would seem of interest then to apply the bounding procedure to the same membrane problem but for the strain-hardening von Mises material, defined by eqns (16) and (48), for the range of strain-hardening index $0 \leqslant m \leqslant 1$, in which convexity prevails. Starting with an upper bound, the explicit expression for the complementary strain energy function reads in this case

$$\bar{W} = \frac{m}{m+1}\sigma_0(\bar{s}/\sigma_0)^{1+1/m} + s_{rz}^2/(2s_{rr}) \tag{58}$$

with the equivalent stress $\bar{s}$ given by eqn (50).

Inserting the trial stress field, eqns $(47)_3$ and $(47)_4$ into inequality (24), via eqns (13) and (58), yields after integration assuming a constant membrane thickness

$$\frac{m(\beta^2-\beta+1)^{(m+1)/(2m)}}{(m+1)(2-\beta-\beta/m)}(1-\kappa^{2-\beta-\beta/m})\frac{a^2t}{\sigma_0^{1/m}}c_2^{(m+1)/m} + \frac{1-\kappa^\beta}{4\pi\beta}\frac{P^2t}{c_2} \geqslant \frac{2m+1}{2(m+1)}Pw(\kappa a) \tag{59}$$

where $w(\kappa a)$ is the deflection at the internal boundary, $r = \kappa a$.

Proceeding by minimizing the left-hand side of inequality (59) with respect to the arbitrary constant c_2, yields

$$2\left\{\left[\frac{(\beta^2-\beta+1)^{1/2}(1-\kappa^\beta)}{2\beta}\right]^{m+1}\left(\frac{1-\kappa^{2-\beta-\beta/m}}{2-\beta-\beta/m}\right)^m\frac{P}{2\pi\sigma_0 at}\right\}^{1/(2m+1)} \geqslant w(\kappa a)/a. \tag{60}$$

Postponing for a moment minimization with respect to the remaining free parameter β, a similar procedure as regards the lower bound in inequality (24), yields when adopting the displacement field, eqns $(47)_1$ and $(47)_2$

$$\frac{2m+1}{2(m+1)}Pw(\kappa a) \geqslant (1-\kappa^\alpha)Pac_1 - \frac{1}{m+1}\left(\frac{\alpha^2}{\sqrt{3}}\right)^{m+1}\cdot\frac{1-\kappa^{2(\alpha m+\alpha-m)}}{\alpha m+\alpha-m}\pi\sigma_0 a^2tc_1^{2(m+1)}. \tag{61}$$

Seeking the maximum of the right-hand side with respect to c_1 then yields

$$w(\kappa a)/a \geqslant \left[\frac{3^{(m+1)/2}(\alpha m+\alpha-m)}{1-\kappa^{2(\alpha m+\alpha-m)}}\left(\frac{1-\kappa^\alpha}{\alpha}\right)^{2(m+1)}\frac{P}{2\pi\sigma_0 at}\right]^{1/(2(m+1))} \tag{62}$$

Furthermore, the right-hand side of inequality (62) achieves its maximum value with respect to α for

Table 1. Upper, $\tilde{w}_U$, and lower, $\tilde{w}_L$, dimensionless deflection bounds for an annular membrane as a function of (strain-hardening index) m; κ is the dimensionless internal radius, and α, β are exponents in trial displacement and stress fields, respectively

κ		0			0.25			0.50			0.75		
m	α	β	$\tilde{w}_L$	$\tilde{w}_U$	β	$\tilde{w}_L$	$\tilde{w}_U$	β	$\tilde{w}_L$	$\tilde{w}_U$	β	$\tilde{w}_L$	$\tilde{w}_U$
0	0	0	∞	∞	0	1.20	1.39	0	0.600	0.693	0	0.249	0.288
0.050	0.0909	0.0911	9.90	10.89	0.205	1.17	1.26	0.352	0.604	0.629	0.460	0.255	0.258
0.100	0.167	0.168	5.57	5.93	0.302	1.15	1.20	0.414	0.608	0.619	0.481	0.261	0.262
0.200	0.287	0.289	3.42	3.50	0.402	1.12	1.13	0.463	0.614	0.617	0.492	0.270	0.270
0.250	0.333	0.337	2.99	3.03	0.430	1.11	1.11	0.475	0.616	0.618	0.495	0.273	0.273
0.300	0.375	0.379	2.70	2.72	0.451	1.10	1.10	0.483	0.618	0.619	0.496	0.276	0.276
0.333	0.400	0.404	2.56	2.57	0.463	1.09	1.09	0.487	0.620	0.620	0.497	0.278	0.278
0.400	0.444	0.447	2.35	2.35	0.481	1.08	1.08	0.493	0.622	0.622	0.498	0.282	0.282
0.500	0.500	0.500	2.14	2.14	0.500	1.07	1.07	0.500	0.625	0.625	0.500	0.286	0.286
1.000	0.667	0.640	1.72	1.73	0.543	1.04	1.04	0.514	0.635	0.636	0.503	0.300	0.300

$$\alpha = \frac{2m}{2m+1}. \tag{63}$$

On introducing the dimensionless deflection

$$\tilde{w}(r/a) = \left(\frac{P}{2\pi\sigma_0 at}\right)^{-1/(2m+1)} w(r)/a \tag{64}$$

then the maximum lower bound, $\tilde{w}_L$ say, is given by

$$\tilde{w}_L(\kappa) = \frac{2m+1}{m}\left(\frac{\sqrt{3}}{4}\right)^{(m+1)/(2m+1)}\left(1-\kappa^{2m/(2m+1)}\right). \tag{65}$$

If σ_0 is replaced by $\sigma_0(\sqrt{3}/2)^{m+1}$ in eqn (64), eqn (65) then coincides with the exact solution for a Tresca material as given by eqns $(47)_2$, (53) and (54).

The associated upper bound, inequality (60), does not lend itself as easily to an analytical search for a minimum. Only for a full membrane ($\kappa = 0$) is it possible to proceed a little further in a sensible way. In this case a dimensionless upper bound $\tilde{w}_U$, in conformity with eqn (64), obtains its minimum value

$$\tilde{w}_U(0) = 2\left\{\left[\frac{(\beta^2-\beta+1)^{1/2}}{2\beta}\right]^{m+1}(2-\beta-\beta/m)^{-m}\right\}^{1/(2m+1)} \tag{66}$$

with β given by

$$\beta^3 - \frac{3m+1}{2m}\beta^2 + \frac{3m+1}{m}\beta - 2 = 0. \tag{67}$$

In the general case it is of course possible to insert for instance the value $\beta = 1/2$, which is known to be exact for $m = 1/2$, into inequality (60), but the upper bound is readily minimized with respect to β by using a simple numerical search technique. Applying such a procedure for selected values of the internal radius, it turns out that the ensuing bounds in general are very close to each other. Thus for $0.25 \leqslant m \leqslant 1$ and arbitrary geometry, the relative difference was always less than 1.5%. As a consequence, it is preferable to present results by explicit numbers and in Table 1 some numerical results are given. In particular, for an incompressible linear elastic solid ($m = 1$), it may be seen that if β is allowed to vary with κa, the internal radius, the upper bound is improved. The relative difference between bounds (56) and (57), based on $\beta = 2/3$, is in this case constant and 1.2% while for instance for the $\kappa = 0.5$ case, the present bounds differ by less than 0.1% (on account of a four figure accuracy not displayed in Table 1).

Static upper bound for deflection at an arbitrary location

As the procedure just illustrated may only be applied to deflections conjugate to an external load agency, in more general circumstances bounds such as inequality (28) have to be resorted to. In case of, for instance, uniform pressure loading, the load-conjugate technique, if applied, will yield bounds for the volume enclosed by deformed and initial membrane shapes (Storåkers, 1983a). In order to illustrate the application of inequality (28), the case of a clamped circular membrane is advantageous as exact solutions, with a numerical accuracy of three figures, are available for a Hookean material (Storåkers, 1983c) and a strain-hardening von Mises material (Hill and Storåkers, 1980).

Thus to bound the transverse deflection at a particular radius of a circular membrane under uniform pressure p by aid of inequality (28), the dummy load system to be applied evidently constitutes of a concentric line load at the particular radius of interest combined with a uniform pressure of magnitude $[p/(m+1)]/2$. It remains then to construct an equilibrated stress field compatible with this loading in order to explicitly compute a value of the bound.

For a start, transverse shear stresses are evidently statically determinate, the only non-vanishing component being

$$s_{rz} = \begin{cases} -\dfrac{pr}{2(m+1)t}, & 0 \leqslant r < \bar{r} \\ -\dfrac{P}{2\pi r t} - \dfrac{pr}{2(m+1)t}, & \bar{r} < r \leqslant a \end{cases} \tag{68}$$

where it is assumed that the concentric line load acts at $r = \bar{r}$, and has a resulting magnitude P.

It may be observed next that the remaining equilibrium equations, eqn (7), are homogeneous and formally independent of the external loading, which facilitates the construction of equilibrated in-plane stresses. It goes without saying though that the accuracy of the resulting bound will depend on the degrees of freedom in the field set up for trial. For reasons of perspicacity, however, the simple field, eqn $(47)_3$, is again adopted and attention confined to the central deflection of a Hookean membrane.

The explicit expression for the complementary strain energy function reads in this case

$$\bar{W} = \frac{1}{2E}(s_{rr}^2 - 2\nu s_{rr}s_{\phi\phi} + s_{\phi\phi}^2) + \frac{1}{2}\frac{s_{rz}^2}{s_{rr}}. \tag{69}$$

Introducing eqn (69) into inequality (29), with stresses given by eqns $(47)_3$ and (68) and setting $m = 1$, $\bar{r} = 0$, yields by using auxiliary notation

$$c_2 = \left(\frac{pa}{Et}\right)^{2/3} Ec, \qquad P = \gamma p \pi a^2,$$

$$w(0)/a \leqslant \left(\frac{pa}{Et}\right)^{1/3}\left[\frac{\beta^2 + 2(1-\nu)(1-\beta)}{2\gamma(1-\beta)}c^2 + \frac{1}{64\gamma c}\left(\frac{16\gamma^2}{\beta} + \frac{8\gamma}{\beta+2} + \frac{1}{\beta+4}\right)\right]. \tag{70}$$

The right-hand side of inequality (70) is then to be minimized with respect to the free parameters β, c, γ. Doing so with respect to c and γ yields

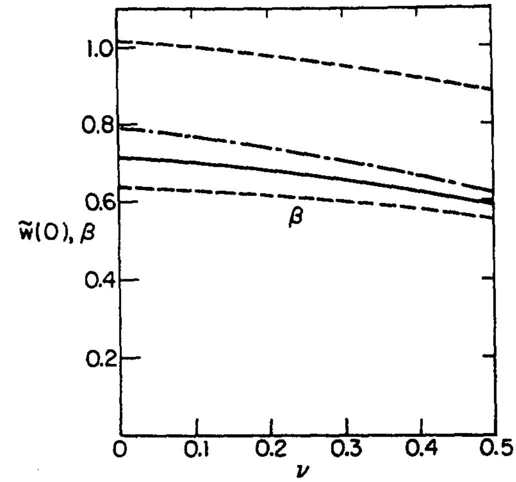

Fig. 1. Dimensionless central deflection, $\tilde{w}(0)$, and stress exponent, β, as a function of Poisson's ratio, ν, for a pressurized circular membrane: ———, exact solution; ----, upper bound; —·—, complementary potential solution.

$$w(0)/a \leqslant \frac{3}{2}\left(\frac{pa}{Et}\right)^{1/3}\left[\frac{\beta^2+2(1-\nu)(1-\beta)}{(1-\beta)(2+\beta)}\right]^{1/3}\frac{[\beta(1+R)+2/(4+\beta)]^{2/3}}{\beta(1+2R)} \tag{71}$$

where

$$R^2 = \frac{(1+\beta)(3+\beta)}{\beta(4+\beta)}.$$

The remaining minimization with respect to β is then readily carried out numerically for different values of ν. The outcome of such a procedure is given in Fig. 1 for the dimensionless deflection

$$\tilde{w}(0) = \left(\frac{pa}{Et}\right)^{-1/3} w(0)/a \tag{72}$$

together with the exact solution.

As may be seen in Fig. 1, although the correct trend as regards Poisson's ratio is reflected by the bound, the accuracy is certainly not of the same order as that found when applying the load-conjugate bounding method. The virtue of the estimate then mainly derives from its property of a safe bound.

The rather poor accuracy achieved may only partly be explained by the simplicity of the stress field utilized. In contrast, adopting the still simpler trial stress field

$$s_{rr} = s_{\phi\phi} = c_2, \qquad s_{rz} = -\frac{pr}{2t} \tag{73}$$

yields via minimization of the complementary potential (13) and subsequent determination of the deflection by eqn (21)

$$w = \left[\frac{(1-\nu)pa}{2Et}\right]^{1/3} a(1-r^2/a^2). \tag{74}$$

The dimensionless central deflection corresponding to eqn (74) is also given in Fig. 1 and as may be seen the inaccuracy is at its worst 11%.

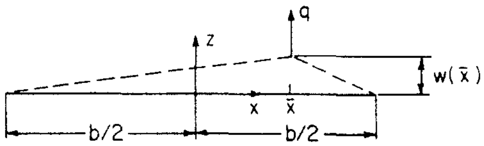

Fig. 2. Section of membrane strip under dummy load.

Dynamic upper bound

Turning finally to the dynamic bound (32), if an exact solution, or as it turns out an upper bound, is known for the deflection under load, $w^*(\bar{x}_\alpha)$, in the static dummy problem, the bound may be directly minimized with respect to the dummy force P.

Thus for a material with a homogeneous complementary strain energy function, by eqn (23) and the self-similarity relation $(35)_1$

$$\int \bar{W}t \, ds = \frac{2m+1}{2(m+1)} P w^*(P, \bar{x}_\alpha) = \frac{2m+1}{2(m+1)} P^{2(m+1)/(2m+1)} \bar{w}^*(\bar{x}_\alpha) \tag{75}$$

in obvious notation.

Introduction of eqn (75) into inequality (32) then yields

$$w(\bar{x}_\alpha) \leqslant \frac{1}{P}\left[K_0 + \frac{2m+1}{2(m+1)} P^{2(m+1)/(2m+1)} \bar{w}^*(\bar{x}_\alpha)\right] \tag{76}$$

and after minimization with respect to P

$$w(\bar{x}_\alpha) \leqslant [2(m+1)K_0/\bar{w}^*(\bar{x}_\alpha)]^{1/(2(m+1))} \bar{w}^*(\bar{x}_\alpha). \tag{77}$$

Thus with the proper interpretation of $\bar{w}^*$, for instance the deflections derived above for annular membranes may be directly inserted into inequality (77).

It is evident though that in general only a statically admissible solution to the dummy problem is required. In order to illustrate a common situation, the case of a membrane strip subjected to an impulse load is considered. For simplicity the strip is assumed to be of infinite length and all variables assumed to be homogeneous in the corresponding direction. It suffices then to introduce a plane Cartesian coordinate system according to Fig. 2.

The dummy line load q per unit length obviously gives rise to shear stresses

$$\left.\begin{aligned} s_{xz} &= \left(\frac{1}{2} - \frac{\bar{x}}{b}\right) q \Big/ t, \qquad -\frac{b}{2} \leqslant x < \bar{x} \\ s_{xz} &= -\left(\frac{1}{2} + \frac{\bar{x}}{b}\right) q \Big/ t, \qquad \bar{x} < x \leqslant \frac{b}{2} \end{aligned}\right\} \tag{78}$$

and homogeneous normal stresses s_{xx}, s_{yy} in the notation of Fig. 2.

For plane strain and a power-law material, inequality (32) then takes the form

$$w(\bar{x}) \leqslant \frac{1}{q}\left\{K_0 + \frac{m\sigma_0}{m+1}\left(\frac{\sqrt{3}s_{xx}}{2\sigma_0}\right)^{(m+1)/m} bt + \left[\left(\frac{1}{2} - \frac{\bar{x}}{b}\right)^2\left(\bar{x} + \frac{b}{2}\right) + \left(\frac{1}{2} + \frac{\bar{x}}{b}\right)^2\left(\frac{b}{2} - \bar{x}\right)\right]\frac{q^2}{2s_{xx}t}\right\} \tag{79}$$

where K_0 is the initial kinetic energy per unit length.

Minimizing the right-hand side of inequality (79) with respect to q and s_{xx} yields

$$w(\bar{x}) = \frac{3^{1/4}}{2}\left[\frac{(m+1)K_0}{\sigma_0 bt}\right]^{1/(2(m+1))}\left[1-\left(\frac{2\bar{x}}{b}\right)^2\right]^{1/2} b. \tag{80}$$

In order to apply this bound to the special case of a Hookean material ($m = 1$), σ_0 in eqn (80) should be replaced by $3E/[4(1-\nu^2)]$. It is perhaps interesting to note though that, irrespective of the value of m, the bounding curve is always an ellipse.

As is evident, the dynamic bound discussed is insensitive to the initial distribution of kinetic energy. Furthermore, as pointed out by Martin in a linear context, the displacements in the static dummy problem and the dynamic problem are governed by different field equations and thus it is not to be expected that the bound may be improved indefinitely. These circumstances consequently exclude the possibility to make any general statements regarding the accuracy of the bound.

It is interesting to note in this context though that Symonds and Mentel (1958) in their analysis of the motion of perfectly plastic beams subjected to impulse loading, as for the present membrane, in the limiting case of a string derived, with some approximation involved, an expression

$$w(0) = \left(\frac{K_0}{2\sigma_0 bt}\right)^{1/2} b \tag{81}$$

for the maximum deflection in the present notation. The same result has also been found by Ploch and Wierzbicki (1981) relying on an approximate method.

Setting $m = 0$ and replacing σ_0 by $\sqrt{3}\sigma_0/2$, as for plane stress, the upper bound according to eqn (80) does in fact coincide with eqn (81) exactly.

CONCLUDING REMARKS

It was shown that in the present formulation of membrane equations, the internal strain energy may be expressed explicitly using combinations of external load potentials. This possibility, which was crucial for the derivation of static bounds, does not seem to prevail in more general circumstances such as for instance when dealing with plates. For plane membranes though, the static bounds derived for classes of elastic and inelastic constitutive relationships, either directly or via correspondence principles, are believed to be applicable to a fairly wide range of real materials.

From a structural point of view the dynamic bound is more general in nature and may be applied to any structure which admits computation of a complementary strain energy or its upper bound. Plane membranes constitute only one member of a large class of slender structures susceptible to deformation at small strains and moderate rotations. Furthermore, in the kinematically linear context, the dynamic bounding technique has earlier been elaborated upon (Ponter, 1975), resulting for instance in time-dependent bounds. Also as no attempt was presently made to account for viscous effects in the dynamic bound, it is believed that there exists more potential to apply bounding techniques to situations where strains are small and rotations moderate.

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